Asymptotic Boundary Layer over a Slender Body of Revolution in Axial Compressible Flow

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The singular perturbation method developed by S. Kaplun is used to solve the thick, compressible laminar boundary layer over a slender body of revolution. The transverse curvature effect is included and is found to be characterized by a dimensionless parameter, which is essentially a function of the ratio of the thickness of the boundary layer to the radius of the body of revolution. An asymptotic series solution is developed for the region where the thickness of the boundary layer is large compared with the radius of the slender body. The nondimensional skin-friction parameter $[(2\pi r_0\tau_w)/(\cos\alpha\ U_e\mu_e)]$ is calculated as a function of the thickness parameter. Interpolations are then made with another known series solution, developed by Probstein and Elliott, which is valid where the boundary layer is thin compared with the radius of the slender body. Thus, interpolation is made to cover the range where the thickness of boundary layer is in the same order as the body radius. From these interpolations, good agreement is obtained between both series for the cone and cylinder. The asymptotic series solution also indicates that the skin friction and heat-transfer rate increase with the Mach number and fall off very slowly in the reciprocal-logarithmic manner with the thickness parameter.

Nomenclature

β = defined by $\beta = S_0\partial U_e/U_e\partial S_0$
C = specific heat or, when not subscripted, Chapman-Rubesin constant
f = defined by $U/U_e = \partial f/\partial\eta$
g = defined by $g = h_s/-[(h_s)_e]$
h = enthalpy
k = coefficient of thermal conductivity of gas
K = defined by Eq. (15)
K_0 = defined by $K_0 = [2\beta(1 + \alpha_0)]/(1 - K)$
l = defined by Eq. (50c)
M = Mach number
P = static pressure
Pr = Prandtl number $C_p\mu/k$
q = local rate of heat transfer from surface per unit area per unit time $q = -k(\partial T/\partial Y)$
r_0 = radius of body of revolution
R = gas constant per gram
γ = ratio of specific heats, $\gamma = C_P/C_V$
s = distance along the body surface, measured from forward stagnation point
S_0 = transformed coordinate along body surface defined by Eq. (6)
T = absolution temperature
u,v = components of velocity parallel and normal to surface
x = distance along body axis
y = distance normal to the body surface
δ = boundary-layer thickness
η = nondimensional coordinate normal to body surface
$\bar{\eta}$ = defined by Eq. (19)
η^* = defined by Eq. (25)
α = angle between the tangent to the meridian profile and the x axis
α_0 = defined by $\alpha_0 = [(r - 1)/2]M_e^2$
μ = absolute viscosity
ρ = gas density
ξ = nondimensional thickness parameter, defined by Eq. (12)
$\bar{\xi}$ = defined by Eq. (19)
ϵ^* = defined by Eq. (25)
Ψ = stream function defined by Eq. (8)
τ = shear stress $\tau = \mu(\partial U/\partial y)$
ϵ = gage function
$\Gamma(n)$ = gamma function

Subscripts

e = quantities outside the boundary layer
W = quantities on the wall surface
∞ = quantities in the freestream
S = quantities at the stagnation point
P = constant pressure
V = constant volume
T = gage function for g
1,2 = gage function for f

Superscripts

$*$ = referring to the inner expansions
$(\bar{\ })$ = referring to the outer expansions
$(\ )'$ = ordinary differentiation with respect to η

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Introduction

THIS paper presents a solution for thick compressible boundary-layer problems using the "method of inner and outer expansions."[3] The development of space vehicles has brought many new aerodynamic problems. Among these is the unusually thick boundary layer, compared to the body radius, that occurs under high temperature conditions. Under such conditions, the curvature of the boundary layer in a plane transverse to the flow has a significant effect on the heat transfer and skin drag. In recent years, solutions for several particular cases have been presented by many earlier works. However, the problem still awaits complete solution for the compressible case. Because of difficulties in treating the problem with dissociation effect, it is assumed in this paper that the gas is not dissociated.

A slender body of revolution is used to investigate the transverse curvature effect. Earlier investigations have found that this effect has been characterized by a dimensionless parameter, which is essentially a function of the ratio of the thickness of the boundary layer to the radius of the body of revolution. This is because the boundary layer must spread out circumferentially. If the boundary layer is

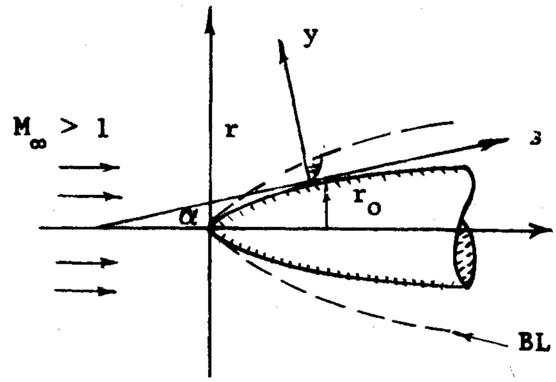

Fig. 1 Steady flow over an unyawed slender body of revolution.

thin, compared with the radius of revolution, and the thickness parameter ξ is less than one, the velocity profile within the boundary layer will be similar to the velocity profile for a flat plate. Therefore, the solution can be found by the perturbation method, using the flat plate solution as the zero order. Seban and Bond[7] obtained the solution for the incompressible case. Later, Probstein and Elliott[2] solved the same problem for compressible flow. Kelly[8] also made important numerical corrections to the Seban-Bond solution. For the region where the thickness of the boundary layer is large, compared with the radius of revolution, the outer circumferential area increases linearly with the thickness of the boundary layer. The problem is therefore different from the two-dimensional flat plate case. The velocity profile within the boundary layer will also differ considerably from that for the flat plate. Glauert and Lighthill[4] showed that, for the incompressible flow, the velocity near the wall is closely proportional to the logarithm of the distance from the centerline of the body. An asymptotic series solution for the large boundary layer, compared to the radius of revolution, was developed by them. Stewartson[5] and Mark[6] also obtained a similar asymptotic series solution valid for the same region. Steiger and Bloom[12] used the integral method to treat the problem including the effect of mass transfer. By employing the integral technique, Raat[13] obtained asymptotic solutions of skin-friction coefficients for an insulated cone in the regions of weak and strong curvature effects.

The purpose of this paper is to present the method of inner and outer expansions as a solution for the compressible case. The method of inner and outer expansions was developed by Kaplun.[3] The application of his theory to this problem follows.

At the outer fringes of the boundary layer, the velocity gradient is very small. We introduce an outer expansion with the freestream velocity as the zero-order solution. However, this expansion will not satisfy the boundary conditions at the wall since the velocity is zero at the wall. Therefore, near the wall an inner expansion is assumed with the zero-order solution equal to zero. The outer and inner solutions are then matched in an intermediate region to solve the undetermined constants by the missing boundary conditions. The skin friction and heat-transfer rate are given by the inner expansions. The complete solutions can be obtained by construction of composite expansions. From the composite expansion, the displacement thickness and momentum thickness can be analytically computed. Since skin friction and heat transfer are of practical interest, the development of the inner and outer expansions are the primary aim of this paper, and the solution of the composite expansion is not included.

Between the results of Probstein's paper for thickness parameter less than one and those developed in the present paper for large thickness parameter lies a gap, which we bridge by interpolations between the two solutions to cover the range where neither solution is applicable. The simplest method of making this interpolation is to plot both solutions against the thickness parameters on a single set of coordinates, and to join both curves with a faired line. The tangent point of this faired line with each curve is a close estimate of the cut off points for the case being investigated. This method is shown in the attached figures.

The thickness parameter cut off point for both solutions varies with Mach number and skin temperature. Consequently, the exact solution of the region between these two cut off points has yet to be developed. However, the interpolation, described in the preceding paragraph, gives a close approximation of the region between the results of Probstein and the method presented here.

Equations of the Boundary Layer

Figure 1 shows the steady flow over an unyawed slender body of revolution. With the coordinate system (s, y) shown in Fig. 1, the basic equations for compressible laminar boundary flow with transverse curvature effect are as follows:

Continuity Equation

$$(\partial/\partial S)(\rho r u) + (\partial/\partial y)(\rho r v) = 0 \tag{1}$$

Momentum Equation

$$\rho\left(u\frac{\partial u}{\partial s} + V\frac{\partial u}{\partial y}\right) + \frac{\partial p_e}{\partial s} = \frac{1}{r}\frac{\partial}{\partial y}\left(r\mu\frac{\partial u}{\partial y}\right) \tag{2}$$

Energy Equation

$$\rho\left(u\frac{\partial hs}{\partial s} + v\frac{\partial hs}{\partial y}\right) = \frac{1\partial}{r\partial y}r\mu \times \left[\left(1 - \frac{1}{Pr}\right)\frac{\partial(u^2/2)}{\partial y} + \frac{1}{Pr}\frac{\partial hs}{\partial y}\right] \tag{3}$$

Equation of State

$$P = \rho RT \tag{4}$$

In order to simplify the calculations, it is customary to assume that 1) Prandtl number is equal to one,

$$Pr = 1 \tag{5a}$$

2) Chapman-Rubesin constant is equal to one,

$$\rho\mu = \rho_e\mu_e \tag{5b}$$

and specific heats are kept constant. It should be noted that Eq. (5b) is a result of the assumption 2) together with the usual boundary-layer approximation that pressure variation across the boundary layer is negligibly small. The Lee's transformation[1] combined with the generalized Mangler transformation for bodies of revolution is introduced:

$$S_0 = \int_0^s \rho_e U_e \mu_e r_0^2 ds \tag{6a}$$

$$\eta = \frac{\rho_e U_e}{(S_0)^{1/2}}\int_0^y r\frac{\rho}{\rho_e}dy \tag{6b}$$

From Eq. (6b) together with the geometrical relation,

$$r(x, y) = r_0(x) + \cos\alpha y \tag{6c}$$

the following useful equation is obtained:

$$\left(\frac{r}{r_0}\right)^2 = 1 + \frac{2\cos(S_0)^{1/2}}{r_0^2\,\rho_e u_e}\int_0^\eta \frac{\rho_e}{\rho}d\eta \tag{6d}$$

Furthermore, a thermal gradient function is also defined as follows:

$$g(\eta, S_0) = hs/(hs)_e \tag{7}$$

A stream function that satisfied the continuity equation is assumed:

$$\Psi = (S_0)^{1/2} f(\eta, S_0) \tag{8}$$

With the aid of Eqs. (5–8), Eqs. (2) and (3) are transformed into the following forms:

$$S_0\left(\frac{\partial f}{\partial \eta}\frac{\partial^2 f}{\partial \eta \partial S_0} - \frac{\partial f}{\partial S_0}\frac{\partial^2 f}{\partial \eta^2}\right) - \frac{\partial^3 f}{\partial \eta^3} - \frac{f\partial^2 f}{2\partial \eta^2} - 2\,\frac{\cos\alpha (S_0)^{1/2}}{u_e \rho_e r_0^2}\,\frac{\partial}{\partial \eta}\left(\frac{\partial^2 f}{\partial \eta^2}\int_0^\eta \frac{\rho_e}{\rho}\,d\eta\right) = \beta\left[\frac{\rho_e}{\rho} - \left(\frac{\partial f}{\partial \eta}\right)^2\right] \tag{9}$$

$$S_0\left(\frac{\partial f}{\partial \eta}\frac{\partial g}{\partial S_0} - \frac{\partial f}{\partial S_0}\frac{\partial g}{\partial \eta}\right) - \frac{f\partial g}{2\partial \eta} - \frac{\partial^2 g}{\partial \eta^2} - 2\,\frac{\cos\alpha (S_0)^{1/2}}{u_e \rho_e r_0^2}\,\frac{\partial}{\partial \eta}\left(\frac{\partial g}{\partial \eta}\int_0^\eta \frac{\rho_e}{\rho}\,d\eta\right) = 0 \tag{10}$$

For the case of a cone or cylinder with an attached shock wave, the value of β is equal to zero. The effect of induced pressure gradient may be neglected to give a good approximate solution. However, under some hypersonic flow conditions,[10, 11] the value of β may be assumed constant for other shapes as well. Therefore, the effect of the pressure gradient is included, but the values of β are assumed to be constant. The density ratio in Eqs. (9) and (10) can be expressed by the following equation:

$$\frac{\rho_e}{\rho} = \frac{T}{T_e} = g + \frac{\gamma - 1}{2} M_e^2\left[g - \left(\frac{\partial f}{\partial \eta}\right)^2\right] \tag{11}$$

which is derived from Eq. (7). From Eqs. (9) and (10), it is seen that a new variable ξ can be defined so that the equations can be transformed into suitable forms with which a similarity solution can be examined:

$$\xi = (\cos\alpha S_0^{1/2})/(\rho_e u_e r_0^2) \tag{12}$$

Equations (9) and (10) expressed in a new coordinate system (ξ, η) will take the following forms:

$$\xi\left(\frac{1}{2} - K\right)\left(\frac{\partial f}{\partial \eta}\frac{\partial^2 f}{\partial \eta \partial \xi} - \frac{\partial f}{\partial \xi}\frac{\partial^2 f}{\partial \eta^2}\right) - \frac{f}{2}\frac{\partial^2 f}{\partial \eta^2} - \frac{\partial^3 f}{\partial \eta^3} - 2\left(1 + \frac{\gamma - 1}{2} M_e^2\right)\xi\frac{\partial}{\partial \eta}\left(\frac{\partial^2 f}{\partial \eta^2}\int_0^\eta g\,d\eta\right) + \frac{\gamma - 1}{2} M_e^2\, 2\xi\frac{\partial}{\partial \eta}\left[\frac{\partial^2 f}{\partial \eta^2}\int_0^\eta \left(\frac{\partial f}{\partial \eta}\right)^2 d\eta\right] = \left(1 + \frac{\gamma - 1}{2} M_e^2\right)\left[g - \left(\frac{\partial f}{\partial \eta}\right)^2\right] \tag{13}$$

$$\xi\left(\frac{1}{2} - K\right)\left(\frac{\partial f}{\partial \eta}\frac{\partial g}{\partial \xi} - \frac{\partial f}{\partial \xi}\frac{\partial g}{\partial \eta}\right) - \frac{f}{2}\frac{\partial g}{\partial \eta} - \frac{\partial^2 g}{\partial \eta^2} - \left(1 + \frac{\gamma - 1}{2} M_e^2\right)2\xi\frac{\partial}{\partial \eta}\left(\frac{\partial g}{\partial \eta}\int_0^\eta g\,d\eta\right) + \frac{(\gamma - 1)}{2} M_e^2\, 2\xi\frac{\partial}{\partial \eta}\left[\frac{\partial g}{\partial \eta}\int_0^\eta \left(\frac{\partial f}{\partial \eta}\right)^2 d\eta\right] = 0 \tag{14}$$

K is a body shape parameter[2] and is defined as

$$K = -\frac{S_0(\rho_e u_e r_0^2)}{\cos\alpha}\,\frac{\partial}{\partial S_0}\,\frac{\cos\alpha}{\rho_e u_e r_0^2} \tag{15}$$

For the case of a cylinder and a cone, the values of ρ_e and u_e are constant along the surface and the values of K are 0 and $\frac{2}{3}$, respectively. For other configurations the value of K lies between 0 and $\frac{2}{3}$ and may not be constant along the surface. This paper considers the cases with constant K values only. Equations (13) and (14) indicate that, when K is equal to $\frac{1}{2}$ and ξ is kept constant, the velocity and thermal gradient functions f and g will depend on η only. Under such conditions, Eqs. (13) and (14) have similarity solutions. This corresponds to the case of a slender body in hypersonic flow with the radius (r_0) and the boundary-layer thickness (δ) varying as $x^{3/4}$ and the induced pressure (P_e) as $x^{-1/2}$. It should be noted that, from Eq. (6d), the dimensionless variable ξ is essentially a function of the ratio of the thickness of the boundary layer to the radius of the body of revolution. The boundary conditions for Eqs. (13) and (14) at the surface of the body are

$$f(0) = (\partial f/\partial \eta)_{\eta=0} = 0 \tag{16a}$$

$$g(0) = g_w \tag{16b}$$

Equation (16a) is due to the requirement of no slip at the surface. Equation (16b) assumes that the surface temperature has been specified. At the outer limits of the boundary layer, measured normal to the surface, the values of u and T are given by the inviscid-flow solution. Thus,

$$(\partial f/\partial \eta)_{\eta=\infty} = 1 \tag{17a}$$

$$g(\infty) = 1 \tag{17b}$$

For small values of ξ, Eqs. (13) and (14) have been solved by Probstein and Elliott.[5]

This paper presents the solution to Eqs. (13) and (14) for large values of ξ, with the parameters β, K, and M_e constant. The corresponding physical flow ranges, where the thickness parameter ξ becomes very large, are the flow regions of a very slender cone and downstream region of a very slender cylinder (or body). For the cone, this results from the fact that the values of S_0/r_0 are very large. For the downstream region, it is due to the large values of x (or S). It should be noted that the actual physical conditions of β, K, and M_e, in general, may not be simultaneously constant. However, for the case of constant wall temperature, the assumed mathematical conditions of constant β, K, and M_e are approximately satisfied in the hypersonic flow region.

Asymptotic Series Solution

From the theory of asymptotic expansions for small Reynolds number,[3] it may be expected that there are two expansions valid in different regions. The outer expansion is valid for the outer limits of the boundary layer and the inner for the region close to the surface. The two expansions will be matched at some intermediate region to determine the constants in each expansion. Large values of ξ correspond, for fixed x, to small values of r_0. However, when r_0 approaches zero, the η coordinate will become infinitely large. For the outer expansion the coordinate system should be chosen in such a fashion that the new coordinate for η will be independent of r_0. Therefore, the following transformations are introduced:

$$\bar{\eta} = \frac{1}{2\xi}\int_0^\eta d\eta \qquad \bar{\xi} = \frac{1}{\xi} \tag{18a}$$

$$\bar{f} = f/2\xi \qquad \bar{g} = g \tag{18b}$$

It should be noted that, when r_0 approaches zero, the new coordinate $\bar{\eta}$ will be kept constant. The transformation of coordinates from η, ξ to $\bar{\eta}$, $\bar{\xi}$ is carried out by means of the relations

$$\frac{\partial}{\partial \eta} = \frac{1}{2\xi}\frac{\partial}{\partial \bar{\eta}} \tag{19a}$$

$$\frac{\partial}{\partial \xi} = \frac{\partial}{\partial \bar{\eta}}\frac{\partial \bar{\eta}}{\partial \xi} - \frac{1}{\xi^2}\frac{\partial}{\partial \bar{\xi}} \tag{19b}$$

Equations (13) and (14) are transformed into the following forms:

$$\bar{\xi}\left(\frac{1}{2}-K\right)\left(\frac{\partial \bar{f}}{\partial \bar{\xi}}\frac{\partial^2 \bar{f}}{\partial \bar{\eta}^2}-\frac{\partial \bar{f}}{\partial \bar{\eta}}\frac{\partial^2 \bar{f}}{\partial \bar{\eta}\partial \bar{\xi}}\right)+(K-1)\bar{f}\frac{\partial^2 \bar{f}}{\partial \bar{\eta}^2}-\frac{\bar{\xi}^2}{4}\frac{\partial^3 \bar{f}}{\partial \bar{\eta}^3}-\left(1+\frac{\gamma-1}{2}M_e^2\right)\left[\frac{\partial}{\partial \bar{\eta}}\left(\frac{\partial^2 \bar{f}}{\partial \bar{\eta}^2}\int_0^{\bar{\eta}}\bar{g}d\bar{\eta}\right)\right]+\frac{\gamma-1}{2}M_e^2\left\{\frac{\partial}{\partial \bar{\eta}}\left[\frac{\partial^2 \bar{f}}{\partial \bar{\eta}^2}\int_0^{\bar{\eta}}\left(\frac{\partial \bar{f}}{\partial \bar{\eta}}\right)^2 d\bar{\eta}\right]\right\}=\beta\left(1+\frac{\gamma-1}{2}M_e^2\right)\left[\bar{g}-\left(\frac{\partial \bar{f}}{\partial \bar{\eta}}\right)^2\right] \quad (20)$$

$$\bar{\xi}\left(\frac{1}{2}-K\right)\left(\frac{\partial \bar{g}}{\partial \bar{\eta}}\frac{\partial \bar{f}}{\partial \bar{\xi}}-\frac{\partial \bar{f}}{\partial \bar{\eta}}\frac{\partial \bar{g}}{\partial \bar{\xi}}\right)+(K-1)\bar{f}\frac{\partial \bar{g}}{\partial \bar{\eta}}-\frac{\bar{\xi}^2}{4}\frac{\partial \bar{g}}{\partial \bar{\eta}}-\left(1+\frac{\gamma-1}{2}M_e^2\right)\left[\frac{\partial}{\partial \bar{\eta}}\left(\frac{\partial \bar{g}}{\partial \bar{\eta}}\int_0^{\bar{\eta}}\bar{g}d\bar{\eta}\right)\right]+\frac{\gamma-1}{2}M_e^2\left\{\frac{\partial}{\partial \bar{\eta}}\left[\frac{\partial \bar{g}}{\partial \bar{\eta}}\int_0^{\bar{\eta}}\left(\frac{\partial \bar{f}}{\partial \bar{\eta}}\right)^2 d\bar{\eta}\right]\right\}=0 \quad (21)$$

At the outer region, the effect of the viscosity term $\partial^3\bar{f}/\partial\bar{\eta}^3$ is very small because there is almost no variation in velocity gradient. From Eq. (20), it can be seen that, for small values of $\bar{\xi}$, this term can be neglected. Furthermore, in the outer region, the zero-order solutions can be assumed to be the free-stream velocity and temperature. The outer expansions can be assumed in the following manner:

$$\frac{\partial \bar{f}}{\partial \bar{\eta}}=1+\epsilon_1(\bar{\xi})\frac{\partial \bar{f}_1}{\partial \bar{\eta}}(\bar{\eta})+\ldots \quad (22a)$$

$$\bar{g}=1+\epsilon_T(\bar{\xi})\bar{g}_1(\bar{\eta})+\ldots \quad (22b)$$

where ϵ_1 and ϵ_T are gage functions, which should approach zero when ξ becomes very large. From Eqs. (20–22b) the following first-order equations are obtained:

$$(1-K)\bar{\eta}\frac{\partial^2 \bar{f}_1}{\partial \bar{\eta}^2}+\frac{\partial}{\partial \bar{\eta}}\left(\bar{\eta}\frac{\partial^2 \bar{f}_1}{\partial \bar{\eta}^2}\right)=\beta\left(1+\frac{\gamma-1}{2}M_e^2\right)\left(2\frac{\partial \bar{f}_1}{\partial \bar{\eta}}-\bar{g}_1\right) \quad (23a)$$

$$(1-K)\bar{\eta}\frac{\partial \bar{g}_1}{\partial \bar{\eta}}+\frac{\partial}{\partial \bar{\eta}}\left(\bar{\eta}\frac{\partial \bar{g}_1}{\partial \bar{\eta}}\right)=0 \quad (23b)$$

The boundary conditions for Eqs. (23a) and (23b) are

$$\bar{f}_1(0)=\left(\frac{\partial \bar{f}_1}{\partial \bar{\eta}}\right)_{\bar{\eta}=0}=0 \qquad \left(\frac{\partial \bar{f}_1}{\partial \bar{\eta}}\right)_{\bar{\eta}=\infty}=0 \quad (23c)$$

$$\bar{g}_1(0)=0 \qquad \bar{g}_1(\infty)=0 \quad (23d)$$

The solution for Eq. (23b) is

$$\bar{g}_1=B_1\int_\infty^{\bar{\eta}}\frac{e^{-(1-K)}}{\bar{\eta}}d\bar{\eta} \quad (24)$$

which satisfied the boundary condition at infinity. The constant B_1 has to be determined by the matching conditions with the inner expansion. The solution for Eq. (23a) will be discussed later.

The inner coordinates are to be chosen such that the new coordinate for η is infinitely large when r_0 approaches zero. This enables the two expansions to match each other at the intermediate region. The inner coordinates are defined as follows:

$$\eta^*=2\xi(\log_e 4\xi^2)^{-1/2}\int_0^\eta d\eta \qquad \xi^*=\xi \quad (25a)$$

$$f^*=2\xi(\log_e 4\xi^2)^{-1/2}f \qquad g^*=(\log_e 4\xi^2)^{1/2}g \quad (25b)$$

The preceding transformations are obtained by first introducing an intermediate solution h_0 with the intermediate coordinate $\eta^{(f)}=\bar{\eta}/f(\xi)$. The solution h_0 is obtained from a proposed equation that is similar to Eq. (30) by dropping all the inertia terms The function $f(\xi)$ is determined from the relation

$$\lim_{\xi\to\infty}|1-h_0|=0 \quad (25c)$$

The most rapid uniform convergence to the Stoke limit[3] is achieved by setting $\eta^*=\eta^{(f)}$. The inner equations for the Stoke region are obtained from Eqs. (13) and (14) as follows:

$$\frac{\partial^3 f^*}{\partial \eta^{*3}}+\left(1+\frac{\gamma-1}{2}M_e^2\right)\left[\frac{\partial}{\partial \eta^*}\left(\frac{\partial^2 f^*}{\partial \eta^{*2}}\int_0^{\eta^*}g^*d\eta^*\right)\right]-\left[(\log_e 4\xi^{*2})^{1/2}\left(\frac{\gamma-1}{2}M_e^2\right)\frac{\partial}{\partial \eta^*}\left(\frac{\partial^2 f^*}{\partial \eta^{*2}}\int_0^{\eta^*}\frac{\partial f^{*2}}{\partial \eta}d\eta^*\right)\right]-\left(\frac{1}{2}-K\right)\frac{\log_e 4\xi^{*2}}{4\xi^{*2}}\left\{\left(\frac{\partial f^*}{\partial \eta^*}\frac{\partial^2 f^*}{\partial \eta^*\partial \xi^*}-\frac{\partial^2 f^*}{\partial \eta^{*2}}\frac{\partial f^*}{\partial \xi^*}\right)+\left(\frac{\partial f^*}{\partial \eta^*}\right)^2[(\log 4\xi^{*2})^{-1}-1]\right\}+\left(\frac{1}{2}-K\right)\frac{f^*}{4\xi^{*2}}\frac{\partial^2 f^*}{\partial \eta^{*2}}+Kf^*\frac{\log_e 4\xi^{*2}}{4\xi^{*2}}\frac{\partial^2 f^*}{\partial \eta^{*2}}=-\beta\left(1+\frac{\gamma-1}{2}M_e^2\right)\times\frac{\log_e 4\xi^{*2}}{4\xi^{*2}}\left[g^*-(\log_e 4\xi^{*2})^{-1/2}\left(\frac{\partial f}{\partial \eta^*}\right)^2\right] \quad (26)$$

$$\frac{\partial^2 g^*}{\partial \eta^{*2}}+\left(1+\frac{\gamma-1}{2}M_e^2\right)\frac{\partial}{\partial \eta^*}\left(\frac{\partial g^*}{\partial \eta^*}\int_0^{\eta^*}g^*d\eta^*\right)-\frac{\gamma-1}{2}M_e^2(\log_e 4\xi^{*2})^{1/2}\frac{\partial}{\partial \eta^*}\left[\frac{\partial g^*}{\partial \eta^*}\int_0^{\eta^*}\left(\frac{\partial f^*}{\partial \eta^*}\right)^2 d\eta^*\right]+\frac{1}{4\xi^{*2}}\left(\frac{1}{2}-K\right)\left(g^*\frac{\partial f^*}{\partial \eta^*}+f^*\frac{\partial g^*}{\partial \eta^*}\right)-\frac{\log_e 4\xi^{*2}}{4\xi^{*2}}\times\left[\left(\frac{1}{2}-K\right)\left(\frac{\partial f^*}{\partial \eta^*}\frac{\partial g^*}{\partial \xi^*}-\frac{\partial f^*}{\partial \xi^*}\frac{\partial g^*}{\partial \eta^*}\right)-Kf^*\frac{\partial g^*}{\partial \eta^*}\right]=0 \quad (27)$$

Since the velocity is zero at the wall but the temperature is not zero, the following expansions are assumed:

$$\frac{\partial f^*}{\partial \eta^*}=0+\epsilon_1^*(\xi^*)\frac{\partial f_1^*(\eta^*)}{\partial \eta^*}+\epsilon_2^*(\xi^*)\frac{\partial f_2^*(\eta^*)}{\partial \eta^*}+\ldots \quad (28a)$$

$$g^*=g_0^*(\eta^*)+\epsilon_T^*(\xi^*)g_1^*(\eta^*)+\ldots \quad (28b)$$

where ϵ_1^* and ϵ_T^* are inner gage functions, which should approach zero when ξ becomes very large. If ϵ_1^* behaves asymptotically for large ξ, as

$$\epsilon_1^*\sim(\log_e 4\xi^2)^{-1/2} \quad (28c)$$

the zero- and first-order equations derived from Eqs. (26–28) are as follows:

$$\frac{\partial^3 f_1^*}{\partial \eta^{*3}}+\left(1+\frac{\gamma-1}{2}M_e^2\right)\frac{\partial}{\partial \eta^*}\left(\frac{\partial^2 f_1^*}{\partial \eta^{*2}}\int_0^{\eta^*}g_0^*d\eta^*\right)=0 \quad (29)$$

$$\frac{\partial^2 g_0^*}{\partial \eta^{*2}}+\left(1+\frac{\gamma-1}{2}M_e^2\right)\frac{\partial}{\partial \eta^*}\left(\frac{\partial g_0^*}{\partial \eta^*}\int_0^{\eta^*}g_0^*d\eta^*\right)=0 \quad (30)$$

with boundary conditions,

$$f_1^*(0)=f_{1_{\eta^*}}{}^{*\prime}(0)=0 \qquad f_{1_{\eta^*}}{}^{*\prime}(\infty)=1 \quad (31a)$$

$$g_0^*(0)=g_w^*=(\log_e 4\xi^2)^{1/2}g_w \qquad g_0^*(\infty)=(\log_e 4\xi^2)^{1/2} \quad (31b)$$

After introducing the inner transformations [Eq. (25)], all of the inertia terms (or transport terms) in Eqs. (26) and (27) become very small and are of the order ξ^{-1} and ξ^{-2}. Furthermore, due to the assumption (28c), all these terms can be neglected for the time being. Later, if we find the assumption (28c) is valid, then these neglected terms are indeed transcendently small. Equation (30) is nonlinear but can be solved in the following manner. By integrating Eq. (30) once, the following equation is obtained:

$$\frac{\partial g_0^*}{\partial \eta^*}\left[1+\left(1+\frac{\gamma-1}{2}M_e^2\right)\int_0^{\eta^*} g_0^* d\eta^*\right] = A_0 \quad (31c)$$

where

$$A_0 = (\partial g_0^*/\partial \eta^*)_{\eta^*=0}$$

If we solve for

$$\int_0^{\eta^*} g_0^* d\eta^*$$

and differentiate once again, the solution to Eq. (31c) is found as follows:

$$A_0\eta^* = \exp\left[-\frac{1}{2A_0}(1+\alpha_0)g_w^{*2}\right]\int_{g_w^*}^{g_0^*} \times \exp\left[\frac{1}{2A_0}(1+\alpha_0)g_0^{*2}dg_0^*\right] \quad (32)$$

It should be noted that Eq. (32) satisfies the boundary condition at the wall. The constant A_0 can be determined by matching conditions with the outer expansion. The matching conditions for Eqs. (22b) and (28b) are as follows. If r_0 approaches zero and $\bar{\eta}$ is fixed, but not equal to zero, ξ will become very large and the following matching conditions can be established:

$$\lim_{\xi\to\infty}[(\bar{g}) - (\log_e 4\xi^2)^{-1/2}g^*] = 0 \quad (33)$$

From Eq. (33), the following matching conditions are obtained:

$$\lim_{\xi\to\infty}[1 - (\log_e 4\xi^2)^{-1/2}(g_0^*)] = 0 \quad (33a)$$

$$\lim_{\xi\to\infty}\left[\frac{[1-(\log_e 4\xi^2)^{-1/2}g_0^*] + \left(\epsilon_T\bar{g}_1 - \dfrac{\epsilon_T^* g_1^*}{(\log_e 4\xi^2)^{1/2}}\right)}{\epsilon_T}\right] = 0 \quad (33b)$$

For large values of ξ, it is seen from Eqs. (25a) and (25b) that η^* and g^* become very large. Equation (32) can be expanded into the following asymptotic relation:

$$(1+\alpha_0)\eta^* = \left(\frac{1}{g_0^*}\right)\times \exp\left\{\frac{(1+\alpha_0)(g_0^{*2}-g_w^{*2})}{2A_0}\right\}\left[1+\frac{A_0}{(1+\alpha_0)}\frac{1}{g_0^{*2}}+\dots\right] \quad (34)$$

Equation (34) can be further reduced to the following form through use of Eqs. (18) and (25) for large values of ξ:

$$g_0^* = (\log_e 4\xi^2)\left(\frac{2A_0}{1+\alpha_0}+g_w^2\right)^{1/2} + \frac{A_0}{(1+\alpha_0)\log_e 4\xi^2}\times\left[\log_e(1+\alpha_0)\bar{\eta}\bar{g}_0 - \frac{A_0}{(\log_e 4\xi^2)(1+\alpha_0)}+\dots\right] \quad (34a)$$

When Eq. (34a) is substituted into Eq. (33a), the following relationship is obtained:

$$A_0 = \frac{(1-g_w^2)\{1+[(\gamma-1)/2]M_e^2\}}{2} \quad (35)$$

The residual of the matching Eq. (33a) is equal to

$$1-(\log_e 4\xi^2)^{-1/2}g_0^* = -\frac{\log_e(1+\alpha_0)}{\log_e 4\xi^2}\left(\frac{1-g_w^2}{2}\right)\bar{\eta} \quad (35a)$$

with $\bar{g}_0 \to 1$ and $\bar{g}_1 \to 0$. Furthermore, Eq. (24) can be expanded into the following form for small values of $\bar{\eta}$[9]:

$$g_1 = B_1[\log_e\bar{\eta} + \log_e r(1-K) + 0(\bar{\eta}) + \dots] \quad (36)$$

where $\log_e r$ is Euler's constant and equal to 0.5772.

By substituting Eq. (36) into (33b) it is found that

$$\epsilon_T = (\log_e 4\xi^2)^{-1} \quad (37a)$$

$$B_1\log_e r(1-K)\bar{\eta} - [(1-g_w^2)/2]\log_e(1+\alpha_0)\bar{\eta} = (\log_e 4\xi^2)^{1/2}\epsilon_T^* g_1^* \quad (37b)$$

Since g_1^* is equal to zero at the wall, the right side of the Eq. (37b) has definite limits for small values of $\bar{\eta}$. In order to cancel the singular term $\log_e\bar{\eta}$ in the left side of Eq. (37b), the values of B_1 are found to be

$$B_1 = [(1-g_w^2)/2] \quad (38)$$

For the first-order momentum equation, Eq. (29) can be reduced to Eq. (30) by the following relation:

$$\partial f_1^*/\partial\eta^* = C_1(g_0^* - g_w^*) \quad (39)$$

This is essentially the Crocco's relation. The constant C_1 can be determined by the following matching condition:

$$\lim_{\xi\to\infty}[1-\epsilon_1^*\partial f_1^*/\partial\eta^*] = 0 \quad (40)$$

By Eq. (34b) it is found that

$$\frac{\partial f_1^*}{\partial\eta^*} = C_1(\log_e 4\xi^2)^{1/2}(1-g_w) + \frac{C_1(1-g_w^2)}{2\log_e 4\xi^2}\log_e(1+\alpha_0)\bar{\eta}+\dots \quad (40a)$$

When substituting Eq. (40a) into (40), the constant C_1 is found to be:

$$C_1 = 1/(1-g_w) \text{ and } \epsilon_1^* = (\log_e 4\xi^2)^{-1/2} \quad (40b)$$

The residual for Eq. (40) is

$$1-\epsilon_1^*\frac{\partial f_1^*}{\partial\eta^*} = -\frac{(1+g_w)}{2\log_e 4\xi^2}[\log_e(1+\alpha_0)\bar{\eta}+\dots] \quad (40c)$$

The solution for Eq. (23a) is

$$\frac{\partial\bar{f}_1}{\partial\bar{\eta}} = D_1\phi + (1-K)\phi\int_0^{\bar{\eta}_k}\frac{\int_0^{\bar{\eta}_k}\bar{\eta}_k e^{\bar{\eta}_k}[F(\bar{\eta}_k)]\phi d\bar{\eta}_k}{\bar{\eta}_k e^{\bar{\eta}_k}\phi^2}d\bar{\eta}_k \quad (41)$$

where

$$\phi = \int_\infty^{\bar{\eta}_k}\frac{(S-\bar{\eta}_k)}{S^{K_0+1}}e^{-s}ds \quad (42)$$

$$F(\bar{\eta}_k) = -\frac{K_0}{2}\frac{(1-g_w^2)}{2}\int_0^{\bar{\eta}_k}\frac{e^{-\bar{\eta}_k}}{\bar{\eta}_k}d\bar{\eta}_k \quad (43)$$

Solution (41) has satisfied the boundary condition at the outer limits because $\phi(\infty) = 0$. The constant D_1 has yet to be determined by the matching condition with the inner expansion as follows: if (41) is expanded into a series convergent for small values of $\bar{\eta}$, the particular solution portion of the complete solution is found to be in the order of $\bar{\eta}\log_e\bar{\eta}$ and the following expansion can be obtained:

$$\frac{\partial\bar{f}_1}{\partial\bar{\eta}} = D_1\left\{\log_e r(1-K)\bar{\eta} + e^{-(1-K)\bar{\eta}}\times\left[\sum_{n=1}^{\infty}(-1)^n\frac{K_0!}{n.n!(K_0-n)!}\right]+0(\bar{\eta})\right\} + \frac{\beta(1+\alpha_0)(1-g_w^2)}{2\Gamma(2)}[(1-K)\bar{\eta}\log_e(1-K)\bar{\eta} + 0(\bar{\eta})+\dots] \quad (44)$$

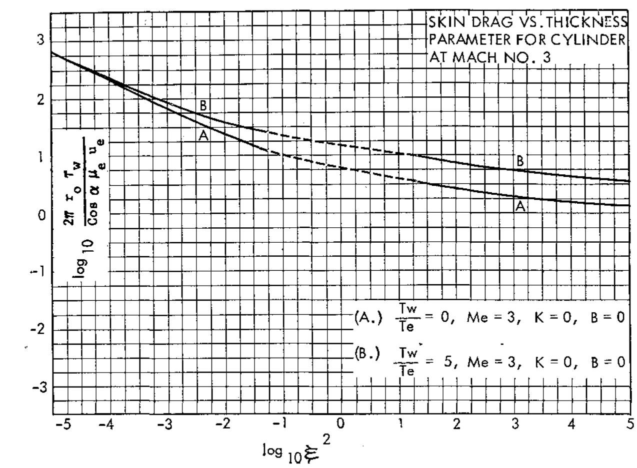

Fig. 2 Skin drag vs thickness parameter for cylinder at Mach number 3.

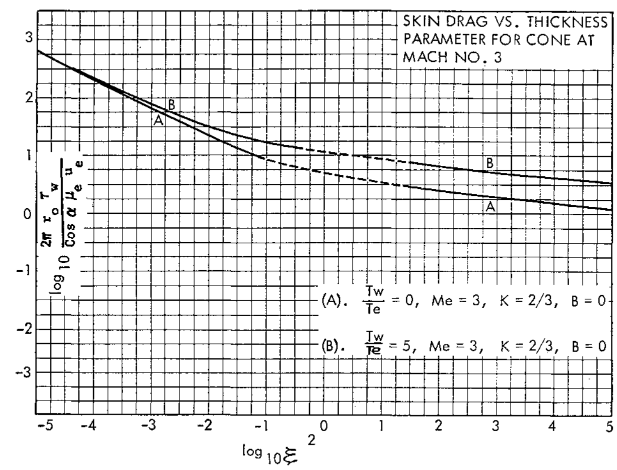

Fig. 3 Skin drag vs thickness parameter for cone at Mach number 3.

The matching condition for $(\partial\bar{f}_1/\partial\eta)$ is as follows:

$$\lim_{\xi\to\infty}\left[\frac{\left[1-\epsilon_1^*\left(\frac{\partial f_1^*}{\partial\eta^*}\right)\right]+\left[\epsilon_1\left(\frac{\partial\bar{f}_1}{\partial\bar{\eta}}\right)-\epsilon_2^*\left(\frac{\partial f_2^*}{\partial\eta^*}\right)\right]}{\epsilon_1}\right]=0 \tag{45}$$

Since $\partial f_2^*/\partial\eta^*$ is zero at the surface, the values of D_1 and ϵ_1 are determined by substituting Eqs. (40c) and (44) into (45) and

$$\epsilon_1 = (\log_e 4\xi^2)^{-1} \tag{46a}$$

$$D_1 = \tfrac{1}{2}(1+g_w) \tag{46b}$$

By substituting (46a) and (46b) into (45), Eq. (45) reduces to:

$$\frac{1+g_w}{2}\left[\log_e\frac{r(1-K)}{1+\alpha_0}+\sum_{n=1}^{\infty}(-1)^{n-1}\frac{K_0!}{n.n!\,(K_0-n)!}\right]=\epsilon_2^*\epsilon_1^{-1}\frac{\partial f_2^*}{\partial\eta^*} \tag{46}$$

The first- and second-order inner equations are as follows:

$$\frac{\partial^3 f_2^*}{\partial\eta^{*3}}+(1+\alpha_0)\frac{\partial}{\partial\eta^*}\left(\frac{\partial^2 f_2^*}{\partial\eta^{*2}}\int_0^{\eta^*}g_0^*d\eta^*+\frac{\partial^2 f_1^*}{\partial\eta^{*2}}\int_0^{\eta^*}g_1^*d\eta^*\right)=0 \tag{47}$$

$$\frac{\partial^2 g_1^*}{\partial\eta^{*2}}+(1+\alpha_0)\frac{\partial}{\partial\eta^*}\left(\frac{\partial g_1^*}{\partial\eta^*}\int_0^{\eta^*}g_0^*d\eta+\frac{\partial g_0^*}{\partial\eta^*}\int_0^{\eta^*}g_1^*d\eta^*\right)=0 \tag{48}$$

The boundary conditions are

$$f_2^*(0)=(\partial f_2^*/\partial\eta^*)_{\eta^*=0}=0 \qquad (\partial f_2^*/\partial\eta^*)_{\eta^*=\infty}=0 \tag{49a}$$

$$g_1^*(0)=0 \qquad g_1^*(\infty)=0 \tag{49b}$$

By integrating Eq. (48) once, we obtain

$$\frac{\partial g_1^*}{\partial\eta^*}+(1+\alpha_0)\left(\frac{\partial g_1}{\partial\eta^*}\int_0^{\eta^*}g_0^*d\eta^*+\frac{\partial g_0^*}{\partial\eta^*}\int_0^{\eta^*}g_1^*d\eta^*\right)=A_1 \tag{50}$$

with $A_1 = g_1^{*\prime}(0)$. By the use of Eq. (31c), (50) can be reduced to

$$\int_0^{\eta^*}g_1^*d\eta^*=(1+\alpha_0)^{-1}\left[\frac{A_1}{\partial g_0^*/\partial\eta^*}-\frac{A_0(\partial g_1^*/\partial\eta^*)}{(\partial g_0^*/\partial\eta^*)^2}\right] \tag{50a}$$

In order to solve the equation, the following substitution is used:

$$g_1^* = l(\eta^*)(\partial g_0^*/\partial\eta^*) \tag{50b}$$

The solution for g_1^* is

$$g_1^*=\frac{A_1}{A_0}g_0^{*\prime}\left(\eta^*-\int\int_0^{\eta^*}\frac{g_0^{*\prime\prime}}{g_0^{*\prime}}d\eta^*d\eta^*\right) \tag{50c}$$

For large values of η^*, the following symptotic relation is obtained:

$$g_1^*=\frac{A_1}{2A_0}\left(\frac{g_0^{*2}-g_w^{*2}}{g_0^*}\right)+\ldots \tag{50d}$$

When Eq. (50d) is substituted into (37b) we obtain

$$\frac{1-g_w^2}{2}\log_e\frac{r(1-K)}{1+\alpha_0}=(\log_e 4\xi^2)\,\epsilon_T^*\left(\frac{A_1}{2A_0}\right)(1-g_w^2) \tag{51}$$

From Eq. (51) it is found that

$$\epsilon_T^* = (\log_e 4\xi^2)^{-1} \tag{52a}$$

$$A_1 = A_0\log_e\frac{r(1-K)}{1+(\gamma-1)/2M_e^2} \tag{52b}$$

For the first-order momentum equation, Eq. (47) is compared with (48), then

$$\frac{\partial f_2^*}{\partial\eta^*}=\frac{1}{1-g_w}(g_1^*) \tag{53}$$

By substituting this into Eq. (46), the following equation is obtained:

$$\frac{1+g_w}{2}\left[\log_e\frac{r(1-K)}{1+\alpha_0}+\sum_{n=1}^{\infty}(-1)^{n-1}\frac{K_0!}{n.n!\,(K_0-n)!}\right]=(\log_e 4\xi^2)^{3/2}\epsilon_2^*\left(\frac{A_2}{2A_0}\right)(1+g_w) \tag{54}$$

The gage function ϵ_2^* is

$$\epsilon_2^*=\frac{\log_e\frac{r(1-K)}{1+\alpha_0}+\sum_{n=1}^{\infty}(-1)^{n-1}\frac{K_0!}{n.n!\,(K_0-n)!}}{(\log_e 4\xi^2)^{3/2}\left(\log_e\frac{r(1-K)}{1+\alpha_0}\right)} \tag{55}$$

The values of $(\partial g/\partial\eta)_{\eta=0}$ and $(\partial^2 f/\partial\eta^2)_{\eta=0}$ take the following forms:

$$\left(\frac{\partial g}{\partial\eta}\right)_{\eta=0}=\frac{(1+\alpha_0)(1-g_w^2)}{\log_e 4\xi^2}\,\xi\left[1+\frac{1}{\log_e 4\xi^2}\times\left(0.5772+\log_e\frac{1-K}{1+\alpha_0}\right)+\ldots\right] \tag{56}$$

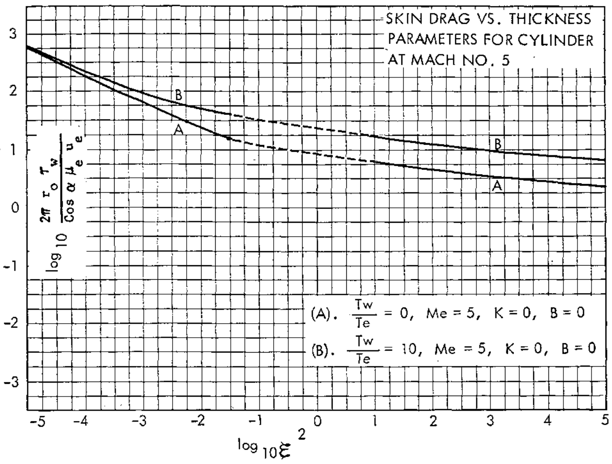

Fig. 4 Skin drag vs thickness parameters for cylinder at Mach number 5.

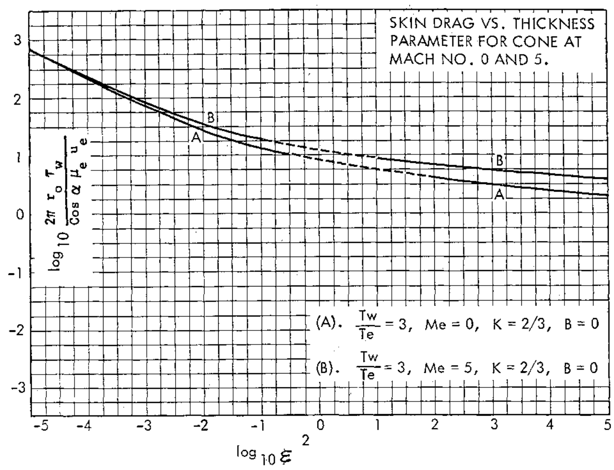

Fig. 5 Skin drag vs thickness parameter for cone at Mach numbers 0 and 5.

$$\left(\frac{\partial^2 f}{\partial \eta^2}\right)_{\eta=0} = \frac{(1+\alpha_0)(1+g_w)}{\log_e 4\xi^2}\,\xi \times \left\{1 + \frac{1}{\log_e 4\xi^2}\times\left[0.5772 + \log_e\frac{1-K}{1+\alpha_0} + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{K_0!}{n.n!\,(K_0-n)!}\right] + \dots\right\} \quad (57)$$

The skin friction and heat-transfer rate are given by:

$$\tau_w = \mu w\left(\frac{\partial u}{\partial Y}\right)_{Y=0} = \frac{\cos\alpha\mu_e u_e}{r_0}\left(\frac{1}{\xi}\right)\left(\frac{\partial^2 f}{\partial \eta^2}\right)_{\eta=0} \quad (58)$$

$$q = -k_w\left(\frac{\partial T}{\partial Y}\right)_{Y=0} = -\frac{\cos\alpha\mu_e hse}{r_0}\left(\frac{1}{\xi}\right)\left(\frac{\partial g}{\partial \eta}\right)_{\eta=0} \quad (59)$$

If $M_e = 0$ and $g_w = 1$, the results can be reduced to the incompressible case. The skin friction, Eq. (58), agrees with the solution given by Glauert and Lighthill for the case of cylinders.

For small values of ξ, Eqs. (9) and (10) can be transformed into the equations given by Probstein with aid of Eq. (11). Therefore, for the region of $\xi < 1$,

$$\tau_w = \frac{\cos\alpha\mu_e u_e}{r_0}\frac{1}{\xi}\,(0.332)(1 + 2A\xi + \dots) \quad (60)$$

for cylinders,

$$A = 0.3420 + 0.7034\,(T_w/T_e) + 0.0866\,(\gamma - 1)M_e^2 \quad (60a)$$

and for cones,

$$A = 0.2583 + 0.4563\,(T_w/T_e) + 0.0603\,(\gamma - 1)M_e^2 \quad (60b)$$

It is noted that the foregoing results are obtained with $\beta = 0$. Since the Crocco's relation still holds for $\beta = 0$ and $P_r = 1$, the heat-transfer rate is obtained as follows:

$$q = \frac{-\cos\alpha\mu_e h_e}{r_0}\left[\left(1 + \frac{\gamma-1}{2}M_e^2\right) - \frac{T_w}{T_e}\right] \times \left(\frac{1}{\xi}\right)(0.332)(1 + 2A\xi + \dots) \quad (61)$$

Results

The frictional drag per unit length of the slender body is given by $F = 2\pi r_0 \tau_w$. For zero pressure gradient, the nondimensional quantity $F/\cos\alpha\mu_e u_e$ is calculated by Eqs. (57) and (60) and is plotted against the thickness parameter ξ as shown in the Figs. 2–6. Figures 2–6 also show the interpolations between the two equations (57) and (60), and demonstrate good conjunction between these two equations. They indicate that, for small values of ξ, the influence of the wall temperature and freestream Mach number on the skin friction and heat-transfer rate becomes extremely small. This results from the fact that, for $\xi < 1$, the zero-order solution is independent of M_e and T_w/T_e.

For the cylinder, Eqs. (58) and (60) can be reduced to the incompressible case by letting $M_e \to 1$ and $g_w \to 1$:

$$\frac{\tau_w r_0}{\mu_e u_e} = \frac{2}{\log_e[(4\mu_e x)/(\rho_e u_e r_0^2)]}\left\{1 + \frac{0.5772}{\log_e[(4\mu_e x)/(\rho_e u_e r_0^2)]} + \dots\right\}$$

$$\frac{\tau_w r_0}{\mu_e u_e} = 0.332\left(\frac{\mu_e x}{\rho_e u_e r_0^2}\right)^{1/2} + 0.6941 + \dots$$

which agree with the leading two terms of the series solutions given by Glauert-Lighthill[4] and Seban-Bond-Kelly.[7, 8]

For the case of a very cold wall, the boundary layer may become very thin and the value of ξ is small. Consequently, the Probstein-Elliot solution, Eq. (60), is essentially valid for

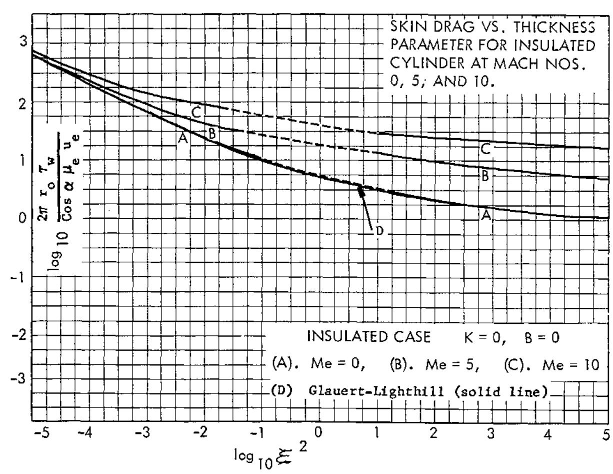

Fig. 6 Skin drag vs thickness parameter for insulated cylinder at Mach numbers 0, 5, and 10.

this case. A similar plot for the heat transfer also can be made. However, by the Crocco's relation,

$$\frac{T}{T_e} = \frac{T_w}{T_e} + \left(1 + \frac{\gamma - 1}{2} M_e^2\right) - \frac{T_w}{T_e} \times \left(\frac{u}{u_e}\right) - \frac{\gamma - 1}{2} M_e^2 \left(\frac{u}{u_e}\right)^2$$

the heat-transfer rate can be calculated from the skin friction and this portion of the numerical presentation is omitted.

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